CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Interpreting Algebraic Expressions

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Beta Version

Interpreting Algebraic Expressions

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to translate between words, symbols, tables, and area representations of algebraic expressions. It will help you to identify and support students who have difficulty in:

- Recognizing the order of algebraic operations.
- Recognizing equivalent expressions.
- Understanding the distributive laws of multiplication and division over addition (expansion of parentheses).

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

A-SSE: Interpret the structure of expressions.

A-APR: Rewrite rational expressions.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

2. Reason abstractly and quantitatively.
7. Look for and make use of structure.

INTRODUCTION

The lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understanding and difficulties. You then review their work, and formulate questions for students to answer, to help them improve their solutions.
- During the lesson, students work in pairs or threes to translate between word, symbol, table of values, and area representations of expressions.
- In a whole-class discussion, students find different representations of expressions and explain their answers.
- Finally, students return to their original assessment task, and try to improve their own responses.

MATERIALS REQUIRED

- Each student will need two copies of the assessment task *Interpreting Expressions* and a mini-whiteboard, pen, and eraser.
- Each pair of students will need a copy of *Card set A: Expressions, Card Set B: Words, Card Set C: Tables, Card Set D: Areas,* a glue stick, a felt-tipped pen, and a large sheet of paper or card for making a poster.
- If you think you will need to continue with the activities into a second lesson, provide envelopes and paper clips for storing matched cards between lessons.
- There are some projector resources to help with instructions.
- The card sets should be cut up before the lesson. Note that the blank cards are part of the activity.

TIME NEEDED

10 minutes for the assessment task, a 1-hour lesson and 10 minutes in a follow-up lesson (or for homework). All timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Interpreting Expressions* (10 minutes)

Have the students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.

Give each student a copy of *Interpreting Expressions*. Introduce the task briefly and help students to understand what they are being asked to do.

I want you to spend ten minutes working individually on this task.

Don't worry too much if you can't understand or do everything. There will be a lesson [tomorrow] with a similar task that will help you improve.

It is important that, as far as possible, students are allowed to answer the questions without assistance.

Interpreting Expressions

1. Write algebraic expressions for each of the following:

   a. Multiply ***n*** by 5 then add 4. ....

   b. Add 4 to ***n*** then multiply by 5. ....

   c. Add 4 to ***n*** then divide by 5. ....

   d. Multiply ***n*** by ***n*** then multiply by 3. ....

   e. Multiply ***n*** by 3 then square the result. ....

2. The equations below were created by students who were asked to write equivalent expressions on either side of the equals sign.

   Imagine you are a teacher. Your job is to decide whether their work is right or wrong. If you see an equation that is false, then:

   a. Cross out the expression on the right and replace it with an expression that is equivalent to the one on the left.

   b. Explain what is wrong, using words or diagrams.

$$2(n + 3) = 2n + 3$$

$$\frac{10n - 5}{5} = 2n - 1$$

$$(5n)^2 = 5n^2$$

$$(n + 3)^2 = n^2 + 3^2 = n^2 + 9$$

If students are struggling to get started, ask them questions that help them understand what is required, but do not do the task for them.

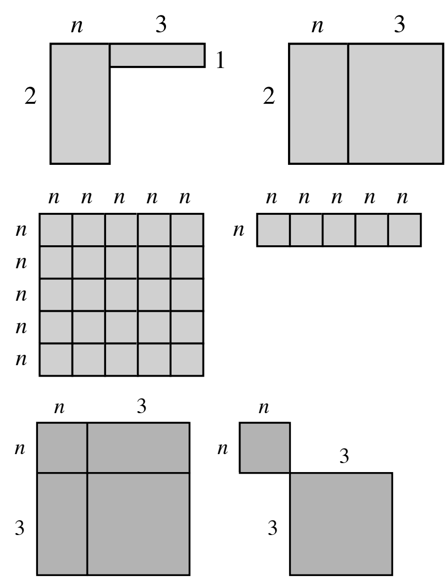
Assessing students' responses

Collect students' responses to the task. Make some notes about what their work reveals about their current levels of understanding. The purpose of doing this is to forewarn you of the difficulties students will experience during the lesson itself, so that you may prepare carefully.

We suggest that you do not score students' papers. The research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a list of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest that you write your own lists of questions, based on your own students' work, using the ideas in the *Common issues* table on the next page. You may choose to write questions on each student's work. If you do not have time to do this, you could write a few questions that will help the majority of students. These can then be displayed on the board at the end of the lesson.

Common issues:	**Suggested questions and prompts:**
Student writes expressions left to right, showing little understanding of the order of operations implied by the symbolic representation. For example: Q1a Writes $n \times 5 + 4$ (not incorrect). Q1b Writes $4 + n \times 5$. Q1c Writes $4 + n \div 5$. Q1d Writes $n \times n \times 3$.	• Can you write answers to the following? $4 + 1 \times 5$ $4 + 2 \times 5$ $4 + 3 \times 5$ • Check your answers with your calculator. How is your calculator working these out? • So what does $4 + n \times 5$ mean? Is this the same as Q1b?
Student does not construct parentheses correctly or expands them incorrectly. For example: Q1b Writes $4 + n \times 5$ instead of $5(n + 4)$. Q1c Writes $4 + n \div 5$ instead of $\frac{4+n}{5}$. Q2 $2(n+3) = 2n+3$ is counted as correct. Q2 $(5n)^2 = 5n^2$ is counted as correct. Q2 $(n+3)^2 = n^2 + 3^2$ is counted as correct.	• Which one of the following is the odd one out and why? • Think of a number, add 3, and then multiply your answer by 2. • Think of a number, multiply it by 2, and then add 3. • Think of a number, multiply it by 2, and then add 6.
Student identifies errors but does not give explanations. In question 2, there are corrections to the first, third, and fourth statements, but no explanation or diagram is used to explain why they are incorrect.	• How would you write down expressions for these areas? • Can you do this in different ways? 

SUGGESTED LESSON OUTLINE

Interactive whole-class introduction (10 minutes)

Give each student a mini-whiteboard, pen and eraser. Hold a short question and answer session. If students show any incorrect answers, write the correct answer on the board and discuss any problems.

On your mini-whiteboards, show me an algebraic expression that means:

Multiply n by 4, and then add 3 to your answer.	$4n+3$
Add 3 to n, and then multiply your answer by 4.	$4(3+n)$
Add 5 to n, and then divide your answer by 3.	$\frac{n+5}{3}$
Multiply n by n, and then multiply your answer by 5.	$5n^2$
Multiply n by 5, and then square your answer.	$(5n)^2$

Collaborative activity: matching expressions and words (15 minutes)

The first activity is designed to help students interpret symbols and realize that the way the symbols are written defines the order of operations.

Organize students into groups of two or three. Display the projector resource P-1, *Matching Expressions and Words*. Note that one of the algebraic expressions on the slide does not have a match in words. This is deliberate! It is to help you explain the task to students.

Model the activity briefly for students, using the examples on the projector resource.

I am going to give each group two sets of cards, one with expressions written in algebra, and the other with words.

Take it in turns to choose an expression and find the words that match it. [4(n + 2) matches 'Add 2 to n then multiply by 4'; 2(n + 4) matches 'Add 4 to n then multiply by 2'.]

When you are working in groups, you should place these cards side by side on the table and explain how you know that they match.

If you cannot find a matching card, then you should write your own. Use the blank cards provided. [4n+2 does not match any of the word cards shown on the slide. The word card 'Multiply n by two, then add four' does not match any of the expressions.]

Give a copy of *Card Set A: Expressions* and *Card Set B: Words* to each small group. Support students in making matches and explaining their decisions.

As they do this, encourage students to speak the algebraic expressions out loud. Students may not be used to 'talking algebra' and may not know how to say what is written, or may do so in ways that create ambiguities.

For example, the following conversation between a teacher and pupil is fairly typical:

Teacher: *Tell me in words what this one says. [Teacher writes:* $3+\frac{n}{2}$*.]*

Pupil: *Three add n divided by two.*

Teacher: *How would you read this one then? [Teacher writes:* $\frac{(3+n)}{2}$ *.]*

Pupil: *Three add n divided by two. Oh, but in the second one you are dividing it all by two.*

Teacher: So can you try reading the first one again, so it sounds different from the second one?

Pupil: Three add… [pause] …n divided by two [said quickly]. Or n divided by two, then add three.

Card Set A: *Expressions*

E1 $\dfrac{n+6}{2}$	E2 $3n^2$
E3 $2n+12$	E4 $2n+6$
E5 $2(n+3)$	E6 $\dfrac{n}{2}+6$
E7 $(3n)^2$	E8 $(n+6)^2$
E9 $n^2+12n+36$	E10 $3+\dfrac{n}{2}$
E11 n^2+6	E12 n^2+6^2
E13	E14

Card Set B: *Words*

W1 Multiply n by two, then add six.	W2 Multiply n by three, then square the answer
W3 Add six to n then multiply by two.	W4 Add six to n then divide by two
W5 Add three to n then multiply by two.	W6 Add six to n then square the answer
W7 Multiply n by two then add twelve	W8 Divide n by two then add six.
W9 Square n, then add six	W10 Square n, then multiply by nine
W11	W12
W13	W14

Students will need to make word cards to match E10: $3+\dfrac{n}{2}$ and E12: n^2+6^2.

They will also need to make expression cards to match W3: *Add 6 to n, then multiply by 2* and W10: *Square n, then multiply by 9*.

Some students may notice that some expressions are equivalent, for example $2(n + 3)$ and $2n + 6$. You do not need to comment on this now: when the *Card Set C: Tables* is given out, students will notice this for themselves.

Collaborative activity: matching expressions, words, and tables (15 minutes)

Card Set C: Tables will make students substitute numbers into the expressions and will alert them to the fact that different expressions are equivalent.

Give each small group of students a copy of *Card Set C: Tables* and ask students to match these to the card sets already on the table. Some tables have numbers missing: students will need to write these in.

Encourage students to use strategies for matching. There are shortcuts that will help to minimize the work. For example, some may notice that:

Card Set C: *Tables*

T1

n	1	2	3	4
Ans	14	16	18	20

T2

n	1	2	3	4
Ans			81	144

T3

n	1	2	3	4
Ans		10	15	22

T4

n	1	2	3	4
Ans	3		27	48

T5

n	1	2	3	4
Ans			81	100

T6

n	1	2	3	4
Ans		10	12	14

T7

n	1	2	3	4
Ans		4		5

T8

n	1	2	3	4
Ans	6.5	7	7.5	8

Since $2(n+3)$ *is an even number, we can just look at tables with even numbers in them.*

Since $(3n)^2$ *is a square number, we can look for tables with only square numbers in them.*

Students will notice that there are fewer tables than expressions. This is because some tables match more than one expression. Allow students time to discover this for themselves. As they do so, encourage them to test that they match for all n. This is the beginning of a generalization.

Do $2(n+3)$ *and* $2n+6$ *always give the same answer when* $n = 1,2,3,4,5$?

What about when $n = 3246$*, or when* $n = -23$*, or when* $n = 0.245$*?*

Check on your calculator.

Can you explain how you can be sure?

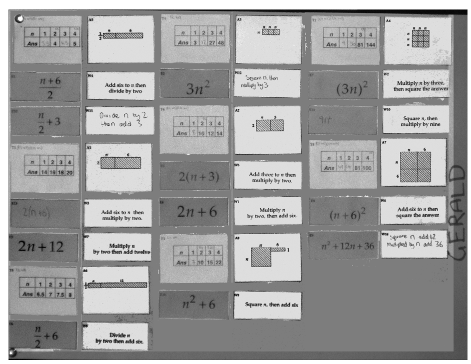
This last question is an important one, and will be followed up in the next part of the lesson.

It is important not to rush the learning. At about this point, some lessons run out of time. If this happens, ask pupils to stack their cards in order, so that matching cards are grouped together, and fasten them with a paper clip. Ask students to write their names on an envelope, and store the matched cards in it. These envelopes can be reissued in the next lesson.

Collaborative activity: matching expressions, words, tables, and areas (15 minutes)

The *Card Set D: Areas* will help students to understand *why* the different expressions match the same tables of numbers.

Give each small group of students a copy of the *Card Set D: Areas*, a large sheet of paper, and a glue stick.



Each of these cards shows an area.

I want you to match these area cards to the cards already on the table.

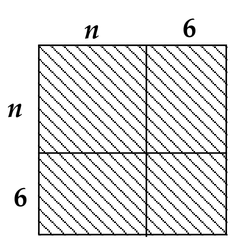
When you reach agreement, paste down your final arrangement onto the large sheet, creating a poster.

Next to each group write down why the areas show that different expressions are equivalent.

These posters will be displayed in the final class discussion.

As students match the cards, encourage them to explain and write down **why** particular pairs of cards go together.

Why does this area correspond to $n^2 + 12n + 36$*?*



Show me where n^2 *is in this diagram. Where is* $12n$*? Where is the* 36 *part of the diagram?*

Now show me why it also shows $(n+6)^2$.

Where is the $n+6$*?*

Ask students to identify groups of expressions that are equivalent and explain their reasoning. For example, E1 is equivalent to E10, E8 is equivalent to E9, and E4 is equivalent to E5.

Whole-class discussion (15 minutes)

Hold a whole-class interactive discussion to review what has been learned over this lesson.

Ask each group of students to justify, using their poster, why two expressions are equivalent.

Then use mini-whiteboards and questioning to begin to generalize the learning.

Draw me an area that shows this expression: $3(x+4)$

Write me a different expression that gives the same area.

Draw me an area that shows this expression: $(4y)^2$

Write me a different expression that gives the same area.

Draw me an area that shows this expression: $(z+5)^2$

Write me a different expression that gives the same area.

Draw me an area that shows this expression: $\dfrac{w+6}{2}$

Write me a different expression that gives the same area.

Review individual solutions to the assessment task (10 minutes)

Return students' work on the assessment task *Interpreting Expressions,* along with a fresh copy of the task sheet. If you chose to write a list of questions rather than write questions on individual papers, display your questions now.

Read through the solution you wrote [yesterday] and think about what you learned this lesson.

Write a new solution, bearing in mind what you've learned, to see if you can improve your work.

If you are running out of time you could postpone this activity until the next lesson, or set it for homework.

SOLUTIONS

This table is for convenience only: it is helpful **not** to refer to cards by these letters in class, but rather to the content of the cards.

Expressions	Words	Tables	Areas
E1	W4		A5
E2		T4	A3
E3	W7	T1	A1
E4	W1	T6	A2
E5	W5		A2
E6	W8	T8	A6
E7	W2	T2	A4
E8	W6	T5	A7
E9	W6		A7
E10		T7	A5
E11	W9	T3	A8
E12			
	W3		
	W10		

Interpreting Expressions

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$$(n + 3)^2 = n^2 + 3^2 = n^2 + 9$$

Card Set A: *Expressions*

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E3 $$2n+12$$	**E4** $$2n+6$$
E5 $$2(n+3)$$	**E6** $$\frac{n}{2}+6$$
E7 $$(3n)^2$$	**E8** $$(n+6)^2$$
E9 $$n^2+12n+36$$	**E10** $$3+\frac{n}{2}$$
E11 $$n^2+6$$	**E12** $$n^2+6^2$$
E13	**E14**

Card Set B: *Words*

W1 Multiply n by two, then add six.	**W2** Multiply n by three, then square the answer.
W3 Add six to n then multiply by two.	**W4** Add six to n then divide by two.
W5 Add three to n then multiply by two.	**W6** Add six to n then square the answer.
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W11	**W12**
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Card Set C: *Tables*

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n	1	2	3	4
Ans		10	12	14

T7

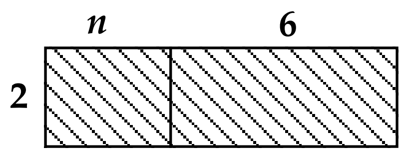
n	1	2	3	4
Ans		4		5

T8

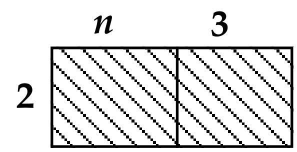
n	1	2	3	4
Ans	6.5	7	7.5	8

Card Set D: *Areas*

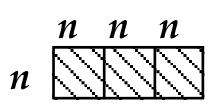
A1



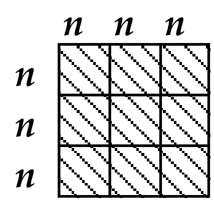
A2



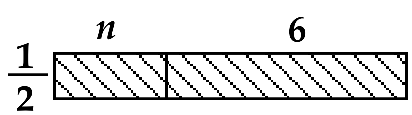
A3



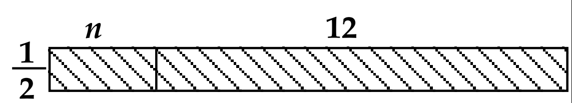
A4



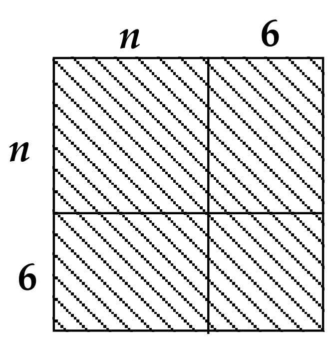
A5



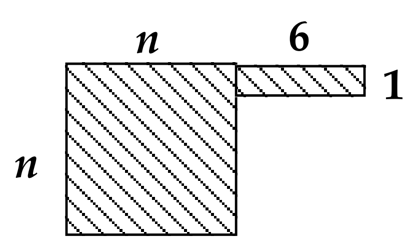
A6



A7



A8



Matching Expressions and Words

$4(n+2)$	Multiply *n* by two, then add four.
$2(n+4)$	Add four to *n*, then multiply by two.
$4n+2$	Add two to *n*, then multiply by four.

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
at the
University of Nottingham
Malcolm Swan, Nichola Clarke, Clare Dawson, Sheila Evans
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

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