



How Student Work Leads to Algebra Success!

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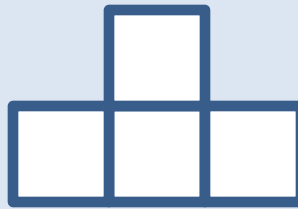


Setting the Stage

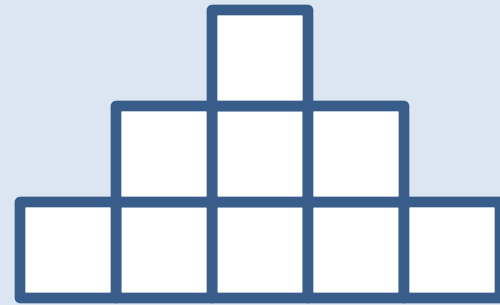
In the figure, as the step changes,
_____ also changes.



Step 1



Step 2



Step 3

Peterson, Blake E. "Linear and Quadratic Change: A Problem from Japan," *Mathematics Teacher*, NCTM: Reston, VA, October 2006. Pages 206-212.





Setting the Stage Solutions

The following list gives some examples of student responses:

perimeter
height
width
size of enclosing rectangle
number of “toothpicks”
number of interior toothpicks
number of intersections
number of corners
number of convex corners
number of squares

number of nonadjacent squares
number of right angles
sum of the interior angles
number of diagonals
leftover space
number of segments
number of parallel lines
length of longest line
number of rectangles

Peterson, Blake E. “Linear and Quadratic Change: A Problem from Japan,”
Mathematics Teacher, NCTM: Reston, VA, October 2006. Pages 206-212.



Problems with Access Points

We will look at some problems that have
“multiple access points.”
Teachers and students can enter at
various points of the problem
based on their
background and readiness.

Contextual problem

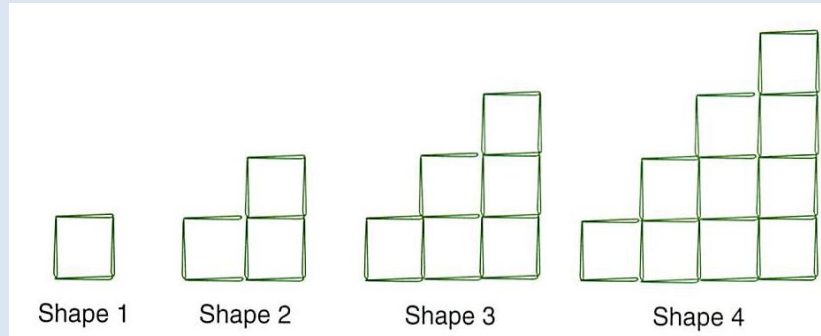
Pre-Algebra

Algebra 1

Algebra 2

Pre-Algebra Access Point

Building with Toothpicks



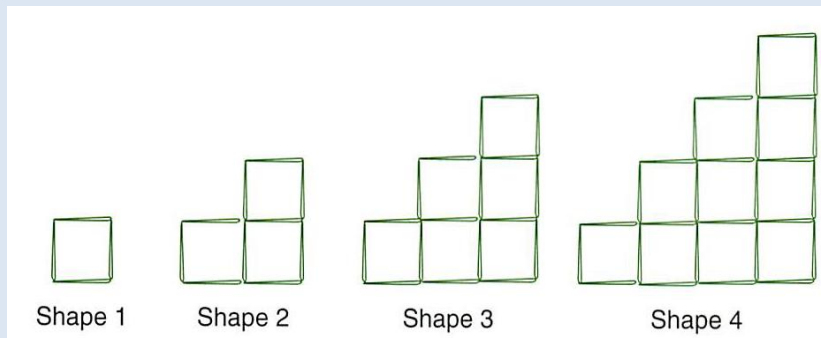
1. Use a pattern from the shapes to determine the perimeter of the fifth shape in the sequence. Show or explain how you arrived at your answer.
2. Write a formula that you could use to find the perimeter of any shape n . Explain how you found your formula.

Friel, Susan, Sid Rachlin, and Dot Doyle. Navigating through Algebra in grades 6-8. National Council of Teachers of Mathematics (NCTM): Reston, VA, 2001.



Algebra 1 Access Point

Building with Toothpicks



3. Create a table and a graph of the first seven shapes in the pattern. What rule did you use to continue the pattern? Explain how you determined your rule.
4. How would the pattern differ if you used triangles instead of squares?

Extension from: Friel, Susan, Sid Rachlin, and Dot Doyle. Navigating through Algebra in grades 6-8. National Council of Teachers of Mathematics (NCTM): Reston, VA, 2001.



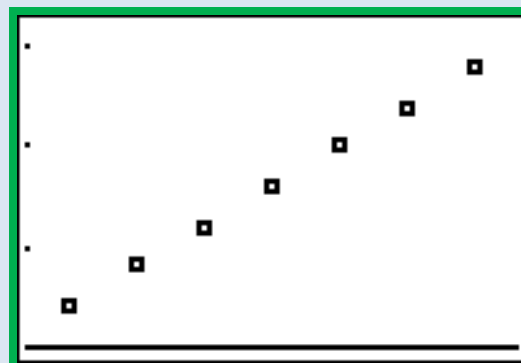


Algebra 1 Access Point

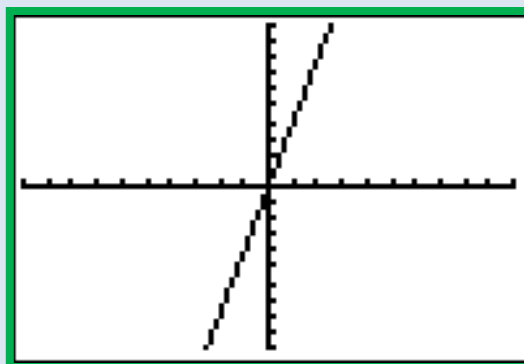
Building with Toothpicks

L1	L2	L3	3
1	4	█	
2	8	██	
3	12	███	
4	16	████	
5	20	█████	
6	24	██████	
7	28	███████	

L3(1)=



Plot1	Plot2	Plot3
Y1	4X	
Y2	=	
Y3	=	
Y4	=	
Y5	=	
Y6	=	
Y7	=	



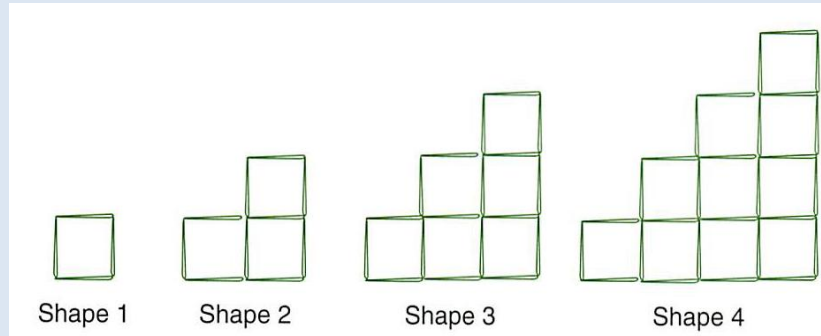
X	Y1
0	0
1	4
2	8
3	12
4	16
5	20
6	24

X=0



Algebra 2 Access Point

Building with Toothpicks



5. Determine the explicit and recursive formulas for finding the perimeter of the n^{th} figure.

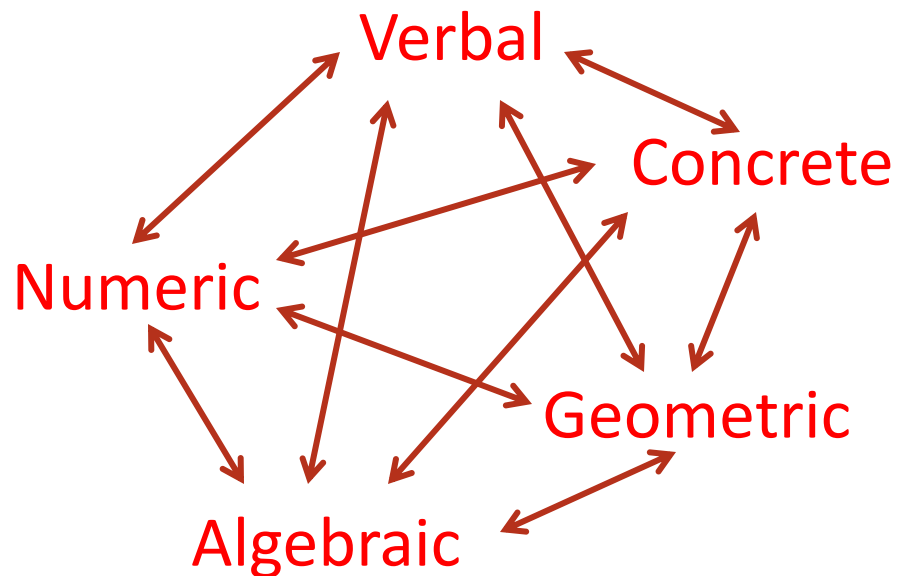
6. What would be the perimeter of the 100^{th} figure?

Extension from: Friel, Susan, Sid Rachlin, and Dot Doyle. Navigating through Algebra in grades 6-8. National Council of Teachers of Mathematics (NCTM): Reston, VA, 2001.



Generalizing

We need to assist our students in moving among representations:



to generalize strategies and solve problems.





Reading and Questioning

The need to read problems carefully is vital – and to learn the importance of re-reading

Determine what is given in the problem and what is being asked

Get students to “unpack” their thinking and ask themselves questions

Ask probing questions to encourage student thinking

Encourage student reflection on their problem solving process





“Chicken/Scale” Problem

Three chickens were weighed in pairs; the first pair weighed in at 10.6 kg, the second pair weighed 8.5 kg, and the third pair weighed 6.1 kg. How much would the scale read if all three chickens were weighed at the same time? How many kilograms does each chicken weigh?



Driscoll, Mark. Fostering Algebraic Thinking, A Guide for Teachers Grades 6-10. Heinemann Publishing: Portsmouth, New Hampshire, 1999.





“Chicken/Scale” Solution

How might a student answer this problem?



“Chicken/Scale” Solution

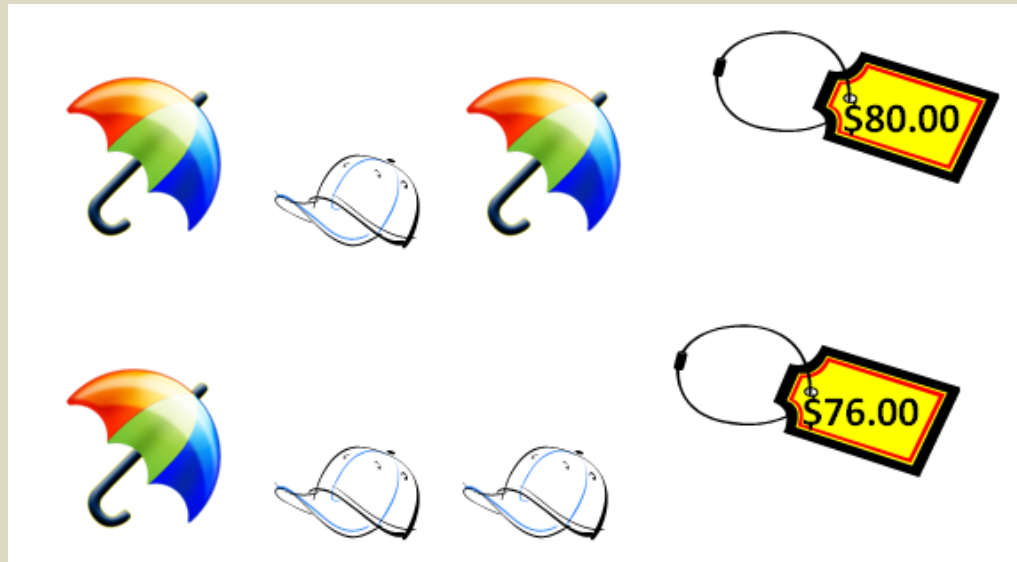
How might a student answer this problem?

Multiple pathways to solution

1. Guess and check
2. Patterns in tables of values
3. System of equations
4. Add three pictures together and divide by two

“Umbrella/Cap” Problem

Finding Prices. The following picture shows the costs of two combinations of umbrellas and hats:



What is the price of one umbrella? One cap?

Driscoll, Mark. Fostering Algebraic Thinking, A Guide for Teachers Grades 6-10. Heinemann Publishing: Portsmouth, New Hampshire, 1999.



“Umbrella/Cap” Problem

What other questions could be asked?





“Umbrella/Cap” Problem

What other questions could be asked?

1. Without calculating the price of each, determine whether the cap or the umbrella is more expensive.
2. What is the difference in price between the cap and the umbrella?
3. Use the two pictures above to make a new combination of umbrellas and caps. Write down the cost of the combination.
4. Make a group of only caps and only umbrellas, then find its price.

Driscoll, Mark. Fostering Algebraic Thinking, A Guide for Teachers Grades 6-10.
Heinemann Publishing: Portsmouth, Hew Hampshire, 1999.





“Umbrella/Cap” Problem

How might a student answer this problem?



“Umbrella/Cap” Problem

Possible umbrella and cap problem solutions

- Cover up a cap and an umbrella from each row and the difference in price between cap and umbrella is \$4.00
- See two rows as pattern and add third row with three caps at \$72. Divide to find 1 cap.
- Same pattern but add row above with three umbrellas at \$84. Divide to find 1 umbrella.
- Create system of equations, $2u + 1c = 80$ and $u + 2c = 76$.
- Might add them together to get $3u + 3c = 156$.

Divide by 3 to get $u + c = 52$

Friedlander, Alex and Michal Tabach, “Promoting Multiple Representations in Algebra” in Cuoco, Albert A. and Frances R. Curcio, editors. The Roles of Representation in School Mathematics. National Council of Teachers of Mathematics (NCTM): Reston, VA, 2001. Problem copyright Encyclopedia Britannica, Inc. 2001. **18**





“Umbrella/Cap” Problem

Combination chart solution:
Follow pattern, then solve

Costs of Combinations (in dollars)

Number of Umbrellas	5						
	4						
	3	84					
	2		80				
	1			76			
	0				72		
		0	1	2	3	4	5
		Number of Caps					

Cuoco, Albert A. and Frances R. Curcio, editors. The Roles of Representation in School Mathematics. National Council of Teachers of Mathematics (NCTM): Reston, VA, 2001.





Patterns of Dots

A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues indefinitely.

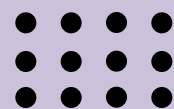
Step 1



Step 2



Step 3



How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.





Solutions?

- What did you do to solve the problem?
- What solution did you get? How can you justify your solution?
- What about the problem allows multiple solutions?





Solutions?

- What did you do to solve the problem?
- What solution did you get? How can you justify your solution?
- What about the problem allows multiple solutions?

Now, let's examine some student work





Examining Student Work

[Sample 1](#)

[Sample 2](#)

[Sample 3](#)

[Sample 4](#)

[Sample 5](#)

[Sample 6](#)





Solution 1

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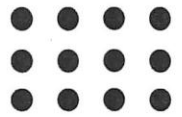
Step 1



Step 2

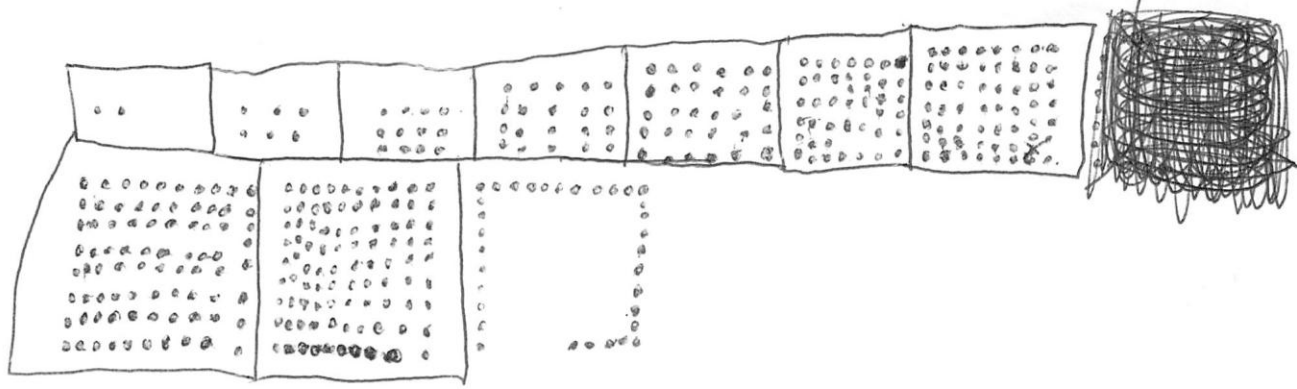


Step 3



How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.





Implications of Solution Type 1

Student started a geometric progression

Next steps could include:

- Asking “What pattern do you notice?”
- Asking “Is there a pattern here? Can you think of a pattern that you could find so that you do not need to draw every step?”
- Move from geometric representation to numeric representation (symbolic).

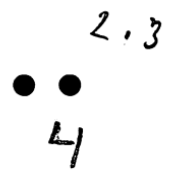


Solution 2

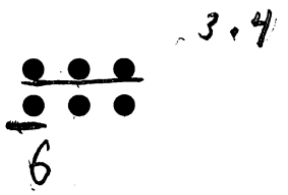
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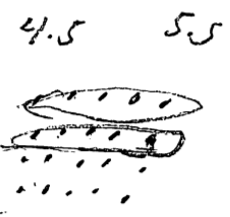
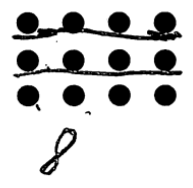
Step 1



Step 2



Step 3



Handwritten numbers: 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42.

How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.

4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42

I think 42 will be the number of dots.





Implications of Solution Type 2

Student found differences but ignored foundational number

Next steps could include:

- Asking “Why did you record 4, 6, 8?”
- Asking “What is the number of dots in steps 1, 2, and 3?”
- Asking if the answers are reasonable.
- Connecting numeric representation to geometric representation.



Solution 3

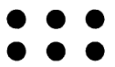
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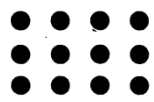
Step 1



Step 2



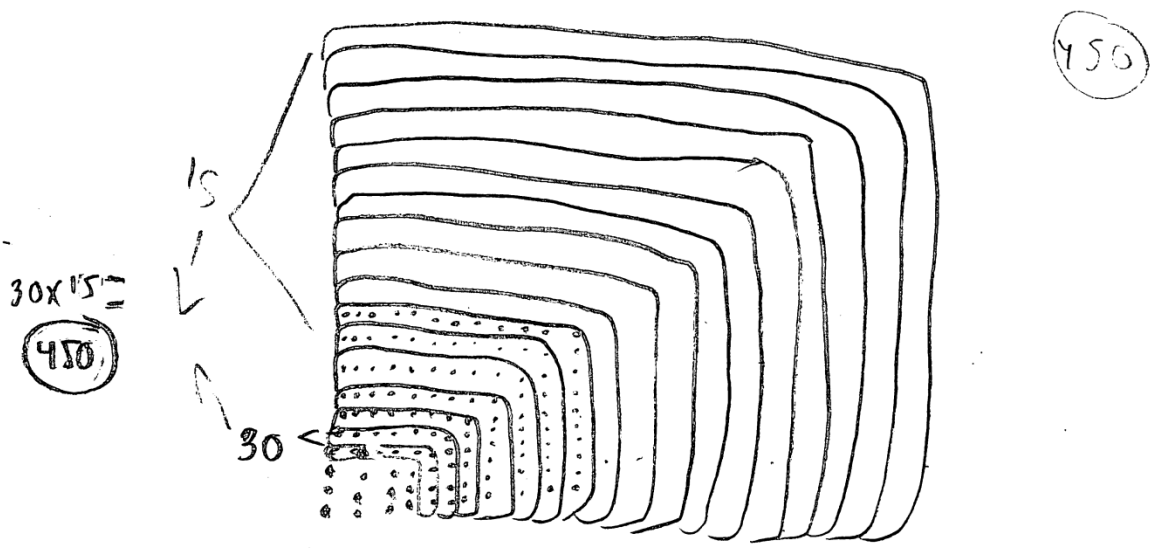
Step 3



How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

by making another shape to match the next

Explain or show how you could do this and give the answer that you get for the number of dots.





Implications of Solution Type 3

Student identifies a geometric pattern

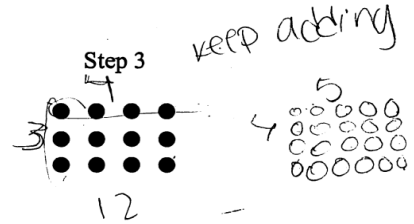
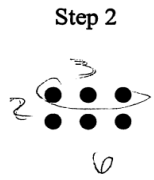
Next steps could include:

- Asking “How did you get 15 and 30?”
- Asking How does 15 and 30 fit with steps 1, 2, and 3?”
- Move from geometric representation to numeric representation (symbolic).



Solution 4

A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues indefinitely.



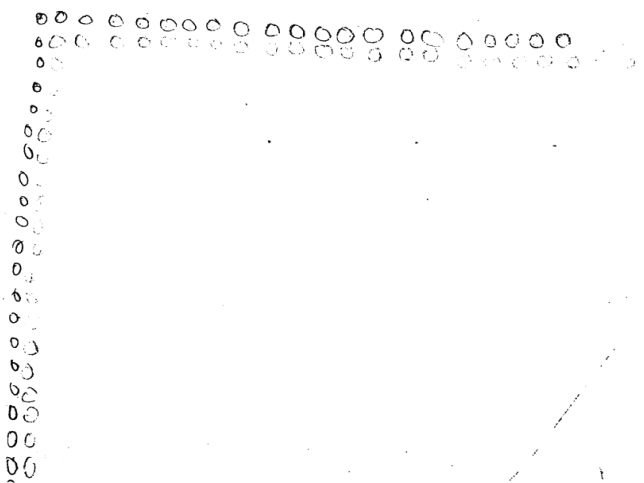
How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.

5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20

5,1 5,6 6,7 7,8 8,9 9,10 10,11 11,12 12,13 13,14 14,15 15,16

16,17 17,18 18,19 19,20 20,21





Implications of Solution Type 4

Student demonstrates geometric and numeric understanding

Next steps could include:

- Asking “What is your actual answer?”
- Asking “Did you check your answer?”
- Move to the next problem



Implications of Solution Type 5

Student found a numeric pattern, which is a repeating pattern instead of a building one, and does not align to the dot progression.

Next steps could include:

- Asking student to show the dot pattern for step 4.
- Asking “Did you determine the 20th step?”
- Connecting numeric representation to geometric representation.



Solution 6

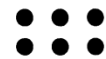
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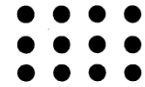
Step 1



Step 2



Step 3



How do you determine the number of dots in Step 20, but not have to draw all 20 pictures and then count the dots?

Explain or show how you could do this and give the answer that you get for the number of dots.

Step 1 has 1 row, step 2 has 2 rows, step 3 has 3 rows, step 4 has 4 rows and so on. Which would mean step 20 would have 20 rows.

5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
30	42	56	72	90	110	132	156	182	210	240	272	306	342	380	420
✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	420





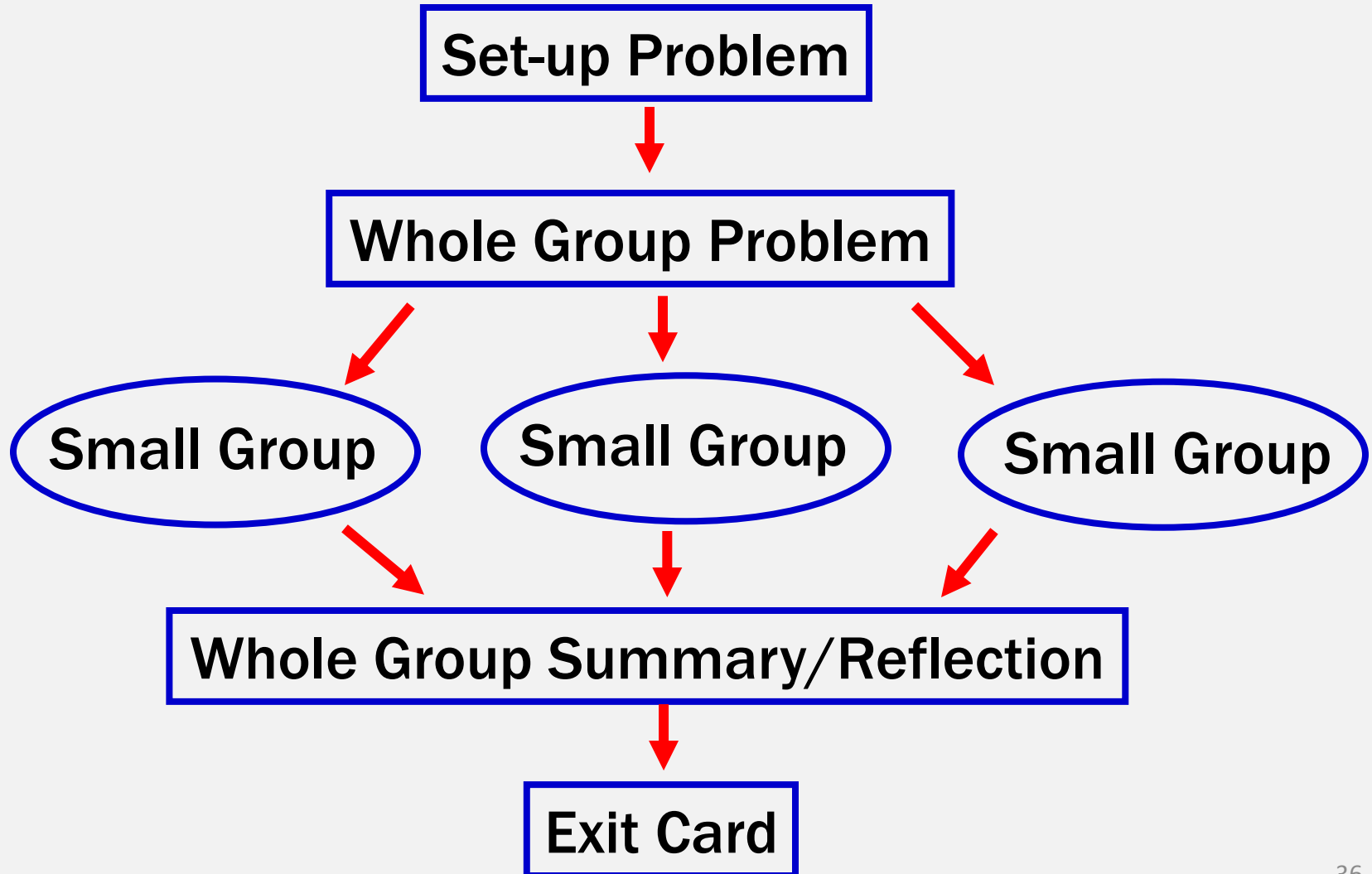
Implications of Solution Type 6

Student demonstrates understanding of differences.

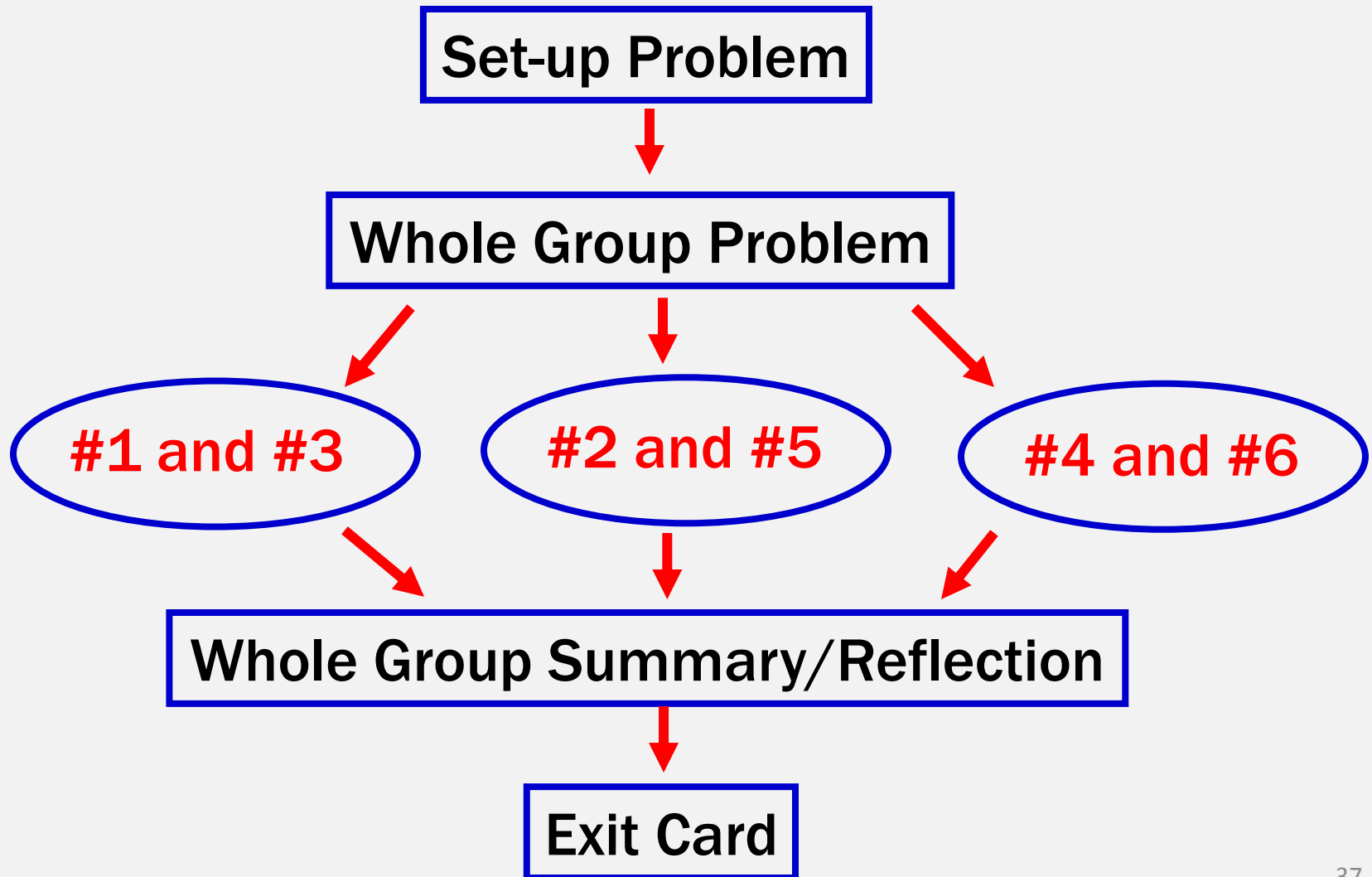
Next steps could include:

- Asking “What would the pattern of dots look like for step 20?”
- Asking “Could you get this answer without determining each step? What about determining the number of dots in step 50?”
- Move to the next problem.

One Possible Lesson Design



One Possible Lesson Design





Other Pattern Problems

Pattern 1 [Perimeter = $2n + 2$]

Shape 1 Shape 2 Shape 3

Pattern 2 [Perimeter = $3n + 2$]

Shape 1 Shape 2 Shape 3

Pattern 3 [Perimeter = $2n + 3$]

Shape 1 Shape 2 Shape 3

Pattern 4 [Area = $2n - 1$]

Shape 1 Shape 2 Shape 3

from Bishop, J, (2000). Linear geometric number patterns: Middle school students' strategies. *Mathematics Education Research Journal*, 12(2), 107-126.





Summary

- What are you walking away with?

Goals:

- Offer activities with multiple access points for students
- Learn how to use student representations
- Interpret student reasoning skills
- Determine the type of instruction that will be best suited for a particular type of learner





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