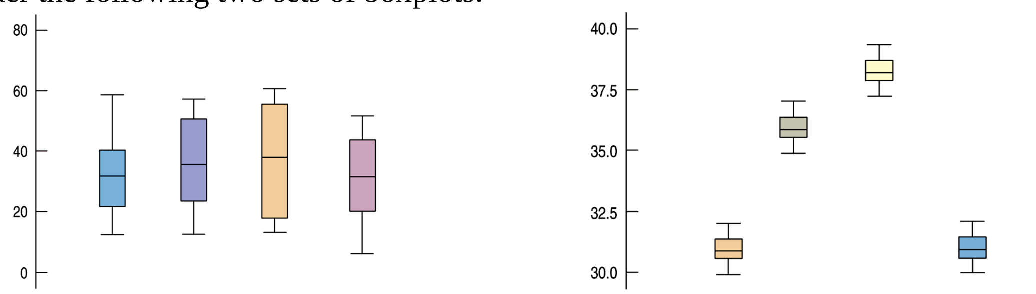
Chapter 28 Summary
Analysis of Variance

What have we learned?

- We can compare the means of more than two independent groups based on samples drawn from those groups.
- We can test the hypothesis that all the means are equal using Analysis of Variance (ANOVA).
    - And, as usual, there are conditions to check.
- When we want to compare pairs of means (after finding significance with an ANOVA *F*-test), we need to use a multiple comparisons method.

Are the Means of Several Groups Equal?

- We already know how to test whether *two* groups have equal means (Chapter 24).
- When we want to test whether more than two groups have equal means, we could compare each pair of groups with a *t*-test.
- However, we'd wind up increasing the probability of a Type I error, since each test would bring with it its own α.
- Fortunately, there is a test that generalizes the *t*-test to *any* number of treatment groups.
- For comparing several means, there is yet another sampling distribution model, called the *F*-model.
- Consider the following two sets of boxplots:



- It's easy to see that the means in the second set differ.
    - It's hard to imagine that the means could be that far apart just from natural sampling variability alone.
- How about the first set? It looks like these observations *could* have occurred from treatments with the same means.
    - This much variation among groups does seem consistent with equal group means.
- Believe it or not, the two sets of treatment means in both figures are the same. (They are 31, 36, 38, and 31, respectively.) Then why do the figures look so different?
- In the second figure, the variation *within* each group is so small that the differences *between* the means stand out.
- This is what we looked for when we compared boxplots by eye back in Chapter 5.
- And it's the central idea of the *F*-test.
    - We compare the differences *between* the means of the groups with the variation *within* the groups.
    - When the differences between means are large compared with the variation within the groups, we reject the null hypothesis and conclude that the means are (probably) not equal.

Are the Means of Several Groups Equal? (cont.)

- In the first figure, the differences among the means look as though they could have arisen just from natural sampling variability from groups with equal means, so there's not enough evidence to reject H_0.
- How can we make this comparison more precise statistically?
- All the tests we've seen have compared differences of some kind with a ruler based on an estimate of variation.
- And we've always done that by looking at the ratio of the statistic to that variation estimate.
- Here, the differences among the means will show up in the numerator, and the ruler we compare them with will be based on the underlying standard deviation—that is, on the variability *within* the treatment groups.

How Different Are They?

- The key to our test will be thinking about the *variation* between groups.
- If the null hypothesis is true, all the treatment means estimate the *same* underlying mean.
    - The means we get for the groups would then vary around the common mean only from natural sampling variation.
    - So, we could act as though the treatment means were just observations and find their variance.
- The more the means resemble each other, the smaller this variance will be; the more they differ, the larger this variance will be.
- How much natural variation should we expect among the means if the null hypothesis is true?
- If the null hypothesis were *true*, then each of the treatment means would estimate the *same* underlying mean.
- We can treat these estimated means as if they were observations and simply calculate their (sample) variance.
    - This variance is the measure we'll use to assess how different the group means are from each other.
    - It's the generalization of the difference between means for only two groups.
- The more the group means resemble each other, the smaller this variance will be. The more they differ (perhaps because the treatments actually have an effect), the larger this variance will be.

The Ruler Within

- We have an estimate from the variation *within* groups. That's traditionally called the error mean square and written MSE.
    - It's just the variance of the residuals.
    - Because it's a pooled variance, we write it s_p^2
- We've got a *separate* estimate from the variation *between* the groups.
    - At least we expect it to estimate *if we assume the null hypothesis is true*. We call this quantity the treatment mean square (MST).

The *F*-Statistic

- When the null hypothesis is true and the treatment means are equal, both MS_E and MS_T estimate σ^2, and their ratio should be close to 1.
- We can use their ratio MS_T/MS_E to test the null hypothesis:
  - If the treatment means really are different, the numerator will tend to be larger than the denominator, and the ratio will be bigger than 1.
- The sampling distribution model for this ratio, found by Sir Ronald Fisher, is called the *F*-distribution. We call the ratio MS_T/MS_E the *F*-statistic.
- By comparing the *F*-statistic to the appropriate *F*-distribution, we (or the computer) can get a P-value.
- The test is one-tailed, because a larger difference in the treatments ultimately leads to a larger *F*-statistic. So the test is significant if the *F*-ratio is "big enough" (and the P-value "small enough").
- The entire analysis is called Analysis of Variance, commonly abbreviated ANOVA.
- Just like Student's *t*, the *F*-models are a family of distributions. However, since we have two variance estimates, we have two degrees of freedom parameters.
- MS_T estimates the variance of the treatment means and has $k - 1$ degrees of freedom when there are k groups.
- MS_E is the pooled estimate of the variance within groups. If there are n observations in each of the k groups, MS_E has $k(n - 1)$ degrees of freedom.

The ANOVA Table

- You'll often see the Mean Squares and other information put into a table called the ANOVA table.
- For the soap example in the book, the ANOVA table is:

Analysis of Variance Table

Source	Sum of Squares	DF	Mean Square	F-ratio	P-value
Soaps	29882	3	9960.64	7.0636	0.0011
Error	39484	28	1410.14		
Total	69366	31			

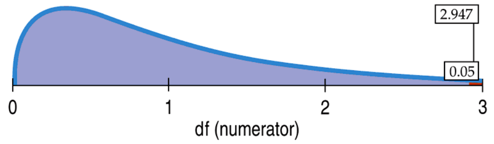
- The ANOVA table was originally designed to hold intermediate calculations so they could easily be repeated.
- With advances in technology, we get all of this information, but we need only look at the *F*-ratio and the P-value.

The *F*-Table

- Usually, you'll get the P-value for the *F*-statistic from technology. Any software program performing an ANOVA will automatically "look up" the appropriate one-sided P-value for the *F*-statistic.
- If you want to do it yourself, you'll need an *F*-table. (On the CD called **Table F.**)
- *F*-tables are usually printed only for a few values of a, often 0.05, 0.01, and 0.001.
- They give the critical value of the *F*-statistic with the appropriate number of degrees of freedom determined by your data, for the alpha-level that you select.
- If your *F*-statistic is greater than that value, you know that its P-value is less than that a level.
- So, you'll be able to tell whether the P-value is greater or less than 0.05, 0.01, or 0.001, but to be more precise, you'll need technology (or an interactive table like the one in *ActivStats*).

The F-Table (cont.)

- Here's an excerpt from an F-table for $\alpha = 0.05$:



df (denominator)	1	2	3	4	5	6	7
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423
25	4.242	3.385	2.991	2.759	2.603	2.490	2.405
26	4.225	3.369	2.975	2.743	2.587	2.474	2.388
27	4.210	3.354	2.960	2.728	2.572	2.459	2.373
28	4.196	3.340	2.947	2.714	2.558	2.445	2.359
29	4.183	3.328	2.934	2.701	2.545	2.432	2.346
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334
31	4.160	3.305	2.911	2.679	2.523	2.409	2.323
32	4.149	3.295	2.901	2.668	2.512	2.399	2.313

The ANOVA Model

- Each observation is the sum of two quantities: the mean of its treatment group and a leftover residual.
- The i-th observation in the k-th group is $y_{ik} = \bar{y}_k + e_{ik}$ where $\bar{y}_k$ is the mean of the k-th group and $e_{ik} = y_{ik} - \bar{y}_k$ is the "error" or residual for the i-th observation in group k.
- The MS_E is the variance of the errors; the MS_T comes from the variance of the group means.
- The ANOVA model gives a fitted or predicted value for each observation as $\hat{y}_{ik} = \bar{y}_k$
  - We predict that each observation will be like its group mean.
- Before we do inference, we must imagine the underlying "true" model for these data.

$$y_{ik} = \mu_k + e_{ik}$$

Back to Standard Deviations

- Variances are easier to work with, but we'd rather have a standard deviation when it's time to think about the data.
- The natural standard deviation to think about is the standard deviation of the residuals.
- The variance of the residuals is MS_E, so the residual standard deviation is

$$s_p = \sqrt{MS_E} = \sqrt{\frac{\sum e^2}{(N-k)}}$$

Assumptions and Conditions

- As in regression, we must perform our checks of assumptions and conditions in order.
- And, as in regression, displays of the residuals are often a good way to check the conditions for ANOVA.

Plot the Data…

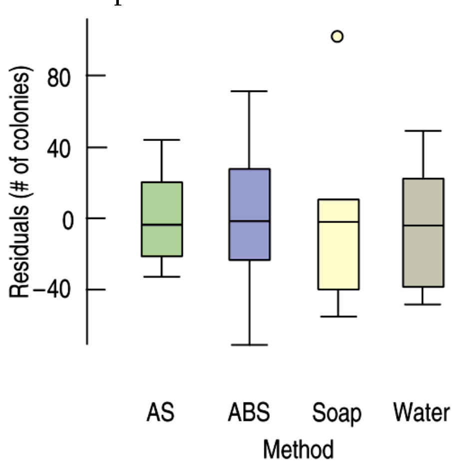
- First examine side-by-side boxplots of the data comparing the responses for all of the groups.
  - Check for outliers within any of the groups (and correct them if there are errors in the data).
  - Get an idea of whether the groups have similar spreads (as we'll need).
  - Get an idea of whether the centers seem to be alike (as the null hypothesis claims) or different.
- If the individual boxplots are all skewed in the same direction, consider re-expressing the response variable to make them more symmetric.

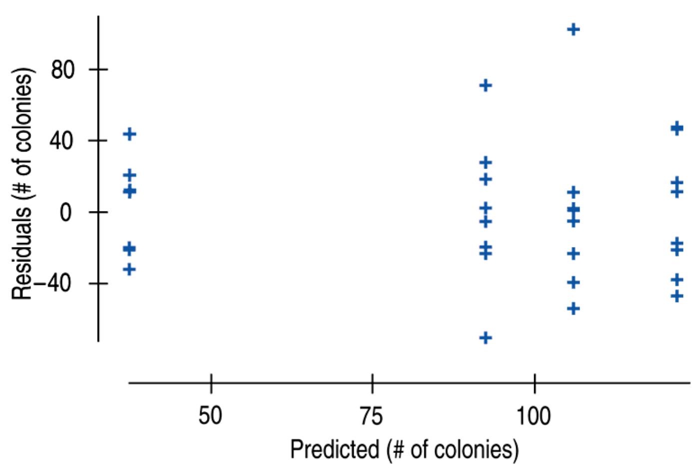
Independence Assumptions

- The groups must be independent of each other.
  - No test can verify this assumption—you have to think about how the data were collected.
- The data *within* each treatment group must be independent as well.
- Check the Randomization Condition: Were the data collected with suitable randomization (a representative random sample or random assignment to treatment groups)?

Equal Variance Assumption

- ANOVA requires that the variances of the treatment groups be equal.
- To check this assumption, we can check that the groups have similar variances:
  - Similar Variance Condition:
    - Look at side-by-side boxplots of the groups to see whether they have roughly the same spread.
    - Look at the original boxplots of the response values again—in general, do the spreads seem to change *systematically* with the centers? (This is more of a problem than random differences in spread among the groups and should not be ignored.)
  - Similar Variance Condition:
    - Look at the residuals plotted against the predicted values. (Larger predicted values lead to larger magnitude residuals, indicating that the condition is violated.)
- In our example, neither of the following plots shows a violation of the equal variance assumption:





Normal Population Assumption

- The F-test requires the underlying errors to follow a Normal Model.
- We will check a corresponding Nearly Normal Condition: examine a histogram of a Normal probability plot of all the residuals together.
- Because we really care about the Normal model *within each group*, the Normal Population Assumption is violated if there are outliers in any of the groups.
  - Check for outliers in the boxplots of the values for each treatment group.

The Balancing Act

- When we have an equal number of cases in each group, this is called balance.
- Experiments that have equal numbers of experimental units in each treatment are said to be balanced or have balanced designs.
- Balanced designs are a bit easier to analyze than unbalanced designs.
    - But in the real world we often encounter unbalanced data.
    - Trust that technology will make the adjustments necessary to analyze unbalanced designs.

Comparing Means

- When we reject H_0, it's natural to ask which means are different.
    - If we can't reject the null, there's nothing more to do.
    - If we've rejected the simple null hypothesis, however, we can test whether any pairs or combinations of group means differ.
- We could do a simple *t*-test about the difference between any pair of means.
    - But, we could ask more complicated questions.

*Bonferroni Multiple Comparisons

- We can't just do multiple simple *t*-tests, since each test poses the risk of a Type I error.
    - As we do more and more tests, the risk that we might make a Type I error grows bigger than the α level of each individual test.
    - If we do enough tests, we're almost sure to reject one of the null hypotheses by mistake—and we'll never know which one.
- To defend against this problem, we will use a method for multiple comparisons.
    - All multiple comparisons methods require that we first reject the overall null hypothesis with the ANOVA's *F*-test.
- The margin of error that we have when testing any pair of means is called the least significant difference (LSD for short): $ME = t^{*} \times s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$
- If two group means differ by more than this amount, then they are significantly different at level α *for each individual test*.
- We still have an issue with examining individual pairs.
- One way to combat this is with the Bonferroni method.
    - This method adjust the LSD to allow for making many comparisons.
    - The result is a wider margin of error called the minimum significant difference (MSD).
    - The MSD is found by replacing t^{*} with a t^{**} that uses a confidence level of (1 - α/J) instead of (1 - α) .

ANOVA on Observational Data

- When ANOVA is used to test equality of group means from observational data, there's no *a priori* reason to think the group variances might be equal at all.
    - Even if the null hypothesis of equal means were true, the groups might easily have different variances.
    - But if the side-by-side boxplots of responses for each group show roughly equal spreads and symmetric, outlier-free distributions, you can use ANOVA on observational data.
    - Be careful, though—if you have not assigned subjects to treatments randomly, you can't draw *causal* conclusions even when the F-test is significant.

What Can Go Wrong?

- Watch out for outliers.
    - One outlier in a group can influence the entire F-test and analysis.
- Watch out for changing variances.
    - If the conditions on the residuals are violated, it may be necessary to re-express the response variable to closer approximate the necessary conditions.
- Be wary of drawing conclusions about causality from observational studies
- Be wary of generalizing to situations other than the one at hand.
- Watch out for multiple comparisons.
    - Use a multiple comparisons method when you want to test many pairs.